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Approximate conditional symmetries for partial differential equations

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Abstract. The method of approximate conditional symmetries for partial differential equations with a small parameter is introduced. We use this method to obtain new approximate solutions of a class of nonlinear wave and heat equations. Also, the notion of a truncated symmetry for a perturbed equation is presented and applied.

1. Introduction

It is well known that the symmetry group method plays an important role in the analysis of differential equations. The original symmetry method for finding symmetry reductions of partial differential equations (PDEs) is the Lie point symmetry group method [1–4]. We generally refer to it as the classical method. It has been successfully applied to find exact solutions and conservation laws of a wide class of PDEs. However, the classical method has some restrictions. Firstly, there are some important nonlinear PDEs, for example the Boussinesq equation [5] and the heat equation with a nonlinear source [6], with very poor Lie point symmetries—they at most admit translations in time and space, and scale transformations. Secondly, some interesting exact solutions such as the multi-soliton solutions and the separable solutions cannot be obtained by the classical method. So the need to extend the classical method arose.

To date, there have been several generalizations of the classical method for symmetry reductions of PDEs, which include the partially invariant solution method due to Ovsiannikov [1], the conditional symmetry method of Bluman and Cole [7] (also referred to as the nonclassical method), the direct method of Clarkson and Kruskal [8], the differential constraint approach of Olver and Rosenau [9] and the generalized conditional symmetry method due to Fokas and Liu, and Zhdanov [10–12]. The conditional symmetry method is similar to the classical method. It consists in augmenting the original PDE with invariant surface conditions, a system of first-order differential equations. The number of determining equations for the conditional symmetry method is generally smaller than the classical method. Although the approaches [6, 13, 24] have been developed to solve the overdetermined system, it is in general difficult to obtain all possible solutions.

Another vital aspect is that many nonlinear PDEs that arise in science and engineering depend on a small parameter. So it is of great importance to find approximate solutions. The ordinary methods for tackling such equations are the numerical and the perturbation methods.

Recently, Baikov, Gazizov and Ibragimov [14–17], in a series of papers, developed the theory and applications of the approximate symmetry group method to find approximate solutions, to calculate approximate conservation laws and approximate Lie–Bäcklund transformation groups of nonlinear DEs.

Consider a kth-order differential equation system $[E]$ which is perturbed up to order p in the small parameter ϵ, namely

$$\begin{aligned} E^\beta(x, u, u_{(1)}, \dots, u_{(k)}; \epsilon) &= E_0^\beta(x, u, u_{(1)}, \dots, u_{(k)}) + \cdots + \epsilon^p E_p^\beta(x, u, u_{(1)}, \dots, u_{(k)}) \\ &= \mathrm{O}(\epsilon^{p+1}) \qquad \beta = 1, \dots, \tilde{m} \end{aligned} \tag{1}$$

where $x = (x^1, x^2, \dots, x^n)$, $u = (u^1, u^2, \dots, u^m)$, E_i^β are smooth functions in their arguments, ϵ a small parameter, $u_{(1)}, u_{(2)}, \dots, u_{(k)}$ are the collection of various order derivatives up to order k, namely, $u_i^\alpha = D_i(u^\alpha)$, $u_{ij}^\alpha = D_j D_i(u^\alpha), \dots,$ being the first and second derivatives, respectively, up to kth order, and

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \cdots \qquad i = 1, \dots, n$$

is the operator of total differentiation with respect to x^i.

We employ the following definition of approximate symmetries of equation (1). The interested reader may refer to [14, 15, 17] and references therein for more details.

An operator $\mathcal{X} = \xi^i(x, u, \epsilon)/\partial x^i + \eta^\alpha(x, u, \epsilon)/\partial u^\alpha$ (summation on i and α is implied) is a pth-order approximate symmetry of equation (1) if

$$\mathcal{X}^{[k]}(E^\beta)|_{E^\beta = \mathrm{O}(\epsilon^{p+1})} = \mathrm{O}(\epsilon^{p+1}) \tag{2}$$

where

$$\begin{aligned} \mathcal{X} &= X_0 + \epsilon X_1 + \cdots + \epsilon^p X_p \\ \mathcal{X}^{[k]} &= X_0^{[k]} + \epsilon X_1^{[k]} + \cdots + \epsilon^p X_p^{[k]} \end{aligned} \tag{3}$$

and

$$\begin{aligned} X_b &= \xi_b^i \frac{\partial}{\partial x^i} + \eta_b^\alpha \frac{\partial}{\partial u^\alpha} \qquad b = 0, \dots, p \\ X_b^{[k]} &= X_b + \zeta_{b,i}^\alpha \frac{\partial}{\partial u_i^\alpha} + \zeta_{b,i_1 i_2}^\alpha \frac{\partial}{\partial u_{i_1 i_2}^\alpha} + \cdots \end{aligned} \tag{4}$$

in which ξ_b^i and η_b^α are functions of (x, u) and the additional coefficients are determined by

$$\zeta_{b,i_1 i_2 \dots i_s}^\alpha = D_{i_1} D_{i_2} \dots D_{i_s}(W_b^\alpha) + \xi_b^j u_{j i_1 i_2 \dots i_s}^\alpha \qquad s = 1, \dots, n \tag{5}$$

where W_b^α is called the Lie characteristic function defined by

$$W_b^\alpha = \eta_b^\alpha - \xi_b^j u_j^\alpha. \tag{6}$$

The X_bs are Lie point symmetry generators. In [14, 15, 17], $X_0 \neq 0$ is called a *stable* symmetry and an *unstable* symmetry otherwise. The *first-order* approximate infinitesimal generators $\mathcal{X} = X_0 + \epsilon X_1$ of a first-order perturbed equation $E^\beta = E_0^\beta + \epsilon E_1^\beta = \mathrm{O}(\epsilon^2)$ can be determined by a three-step algorithm (see [17]).

The purpose of the present paper is to study the approximate conditional symmetries of PDEs. In section 2, we discuss the method of approximate conditional symmetry. In sections 3 and 4, we use the method of section 2 to obtain the approximate conditional symmetries and approximate conditional invariant solutions for nonlinear wave and heat equations. Section 5 contains a discussion of our results.

2. Approximate conditional symmetry method

The approximate conditional symmetry approach is a natural generalization of the approximate classical symmetry method in the same way as the conditional symmetry method is a generalization of the classical symmetry method. Consider the perturbed equation (1). Suppose (1) admits the approximate symmetry generated by $\mathcal{X}$. Then the solution

$$u^\alpha = \sum_{b=0}^{p} \epsilon^b u_b^\alpha(x) \tag{7}$$

of equation (1) is an approximate invariant solution of (1) under a one-parameter subgroup generated by $\mathcal{X}$ if the invariant surface conditions

$$\sum_{b=0}^{p} \epsilon^b W_b^\alpha = \sum_{b=0}^{p} \epsilon^b (\eta_b^\alpha - \xi_b^i u_i^\alpha) = \mathrm{O}(\epsilon^{p+1}) \tag{8}$$

holds together with equation (1). Now let us denote by $[W]$ the surface given by the following system:

$$\sum_{b=0}^{p} \epsilon^b W_b^\alpha = \mathrm{O}(\epsilon^{p+1}) \qquad \partial_{i_1} \ldots \partial_{i_s}\left[\sum_{b=0}^{p} \epsilon^b W_b^\alpha\right] = \mathrm{O}(\epsilon^{p+1}) \qquad s = 1, \ldots, k-1 \tag{9}$$

where $\partial_{i_s} = \partial/\partial x^{i_s}$, $i_s = 1, \ldots, n$. This set of surface conditions are just the $(k-1)$th prolongations of the invariant surface conditions $\sum_{b=0}^{p} \epsilon^b W_b^\alpha = \mathrm{O}(\epsilon^{p+1})$.

The graph of the approximate solutions defines a submanifold of the space of independent and dependent variables. The solution will be invariant under an approximate one-parameter group generated by $\mathcal{X}$ in the pth order of precision if the submanifold is an invariant submanifold of this group in the pth order of precision. This solution is obtained by solving the invariance surface conditions (8) together with equation (1). For equations (1) and (9) to be compatible, the kth prolongation $\mathcal{X}^{[k]}$ of the generator must be a tangent to the intersection of the manifold $[E]$ and the surface $[W]$ in the pth order of precision, i.e. $\mathcal{X}^{[k]}(E^\beta)|_{[W]\cap[E]} = \mathrm{O}(\epsilon^{p+1})$.

Definition 1. *The operator $\mathcal{X}$ is said to be an approximate conditional symmetry generator, of order p, of equation (1) if*

$$\mathcal{X}^{[k]}(E^\beta)|_{[W]\cap[E]} = \mathrm{O}(\epsilon^{p+1}). \tag{10}$$

Equation (10) is a natural extension of (2). The significant difference is that $X_0^{[k]}(E_0^\beta)|_{[W]}$ in (10) contains the ϵ^j-terms, $j = 0, \ldots, p$, whereas $X_0(E_0^\beta)|_{E_0^\beta=0}$ in the usual method does not depend on ϵ. From equation (10), we obtain a nonlinear overdetermined system for ξ_b^i and η_b^α. So the approximate conditional symmetries generally does not form an approximate vector space, i.e. the Lie bracket of any two approximate conditional symmetries is in general not an approximate conditional symmetry. Another important feature is that if $\mathcal{X}$ is an approximate conditional symmetry operator, for any arbitrary function $\lambda(x, u, \epsilon)$, $\lambda\mathcal{X}$ is also an approximate conditional symmetry operator. In fact, as for the nonclassical method, we have that if $\mathcal{X}$ is an approximate conditional symmetry, then so is $\lambda\mathcal{X}$ giving rise to the same invariant surface conditions (8). This property allows us to normalize one of the nonvanishing coefficients in the operator by taking it to be one.

We have, similar to the case of the approximate classical symmetry, the following result.

Theorem 1. *Suppose $\mathcal{X}$ is an approximate conditional symmetry of equation (1), then X_0 is a conditional symmetry of the unperturbed equation (1), namely of $E_0^\beta = 0$.*

Such an exact conditional symmetry X_0 is called a *stable* conditional symmetry of the unperturbed equation (7).

Definition 2. *Suppose that $u = f_0(x)$ is a conditional invariant solution of a conditional symmetry X_0 of the unperturbed equation $E_0^\beta = 0$ and that $u = f_0(x, \epsilon)$ is an approximate conditional invariant solution of the approximate conditional symmetry $\mathcal{X}$. If $\lim_{\epsilon \to 0} f(x, \epsilon) = f_0(x)$, then $u = f_0(x)$ is called a stable solution of the unperturbed equation $E_0^\beta = 0$. Otherwise $u = f_0(x)$ is said to be an unstable solution.*

For the perturbed equation (1), it is also of great importance to obtain the exact solutions. We expand the infinitesimals $\xi^i(x, u, \epsilon)$ and $\eta^\alpha(x, u, \epsilon)$ about ϵ into series as

$$\begin{aligned} \xi^i &= \sum_{l=0}^{\infty} \xi_l^i(x, u)\epsilon^l \\ \eta^\alpha &= \sum_{l=0}^{\infty} \eta_l^\alpha(x, u)\epsilon^l. \end{aligned} \tag{11}$$

Definition 3. *If there exists a positive integer N such that for $l > N$ in (11),*

$$\xi_l^i = \eta_l^\alpha = 0$$

then the exact symmetry

$$\mathcal{X} = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha}$$

of a given perturbed equation (1), is called an Nth-order truncated conditional symmetry.

3. Nonlinear wave equation

We shall consider the (1 + 1)-dimensional case in our applications; that is, one dependent variable u and the independent variables t and x. Also we take $p = 1$ and $\mathcal{X}$ to be the generator of a point symmetry in the form

$$\begin{aligned} \mathcal{X} = {} & (\tau_0(t, x, u) + \epsilon\tau_1(t, x, u))\frac{\partial}{\partial t} + (\xi_0(t, x, u) + \epsilon\xi_1(t, x, u))\frac{\partial}{\partial x} \\ & + (\phi_0(t, x, u) + \epsilon\phi_1(t, x, u))\frac{\partial}{\partial u}. \end{aligned} \tag{12}$$

We can split the operator $\mathcal{X}$ into two cases:

(a) $\mathcal{X}$ with $\tau_0 \neq 0$ and we choose $\tau_0 = 1$, $\tau_1 = 0$.
(b) $\mathcal{X}$ with $\tau_0 = 0$ and we select $\xi_0 = 1$ and $\xi_1 = 0$.

In this section, we discuss approximate conditional symmetries and approximate conditional invariant solutions of the perturbed nonlinear wave equation

$$u_{tt} + \epsilon u_t = (uu_x)_x \tag{13}$$

which arises from one-dimensional gas dynamics [18] and longitudinal wave propagation on a moving threadline [19]. The classical and conditional symmetries of the unperturbed equation of equation (13) were, respectively, discussed in [18, 20]. The approximate classical symmetries of equation (13) was discussed by Baikov *et al* [15, 17]. Following section 2, we look for the approximate conditional symmetries of the form (12). We distinguish two cases:

Case 1. $\tau_0 \neq 0,\ \tau_0 = 1,\ \tau_1 = 0.$

From (10), the system of determining equations for ξ_1, ϕ_1 is

$$\begin{aligned}
\big[&\phi_{1tt} + 2\phi_0\phi_{1tu} + \phi_0^2\phi_{1uu} - 2\xi_{1t}\phi_{0x} - 2\phi_0\xi_{1u}\phi_{0x} + \phi_{0t} - 2\xi_{0t}\phi_{1x} - u\phi_{1xx} + 2\phi_1\phi_{0tu}\\
&+\big(u-\xi_0^2\big)^{-1}(2\xi_0\xi_{0t} - \phi_0 + 2u\xi_{0x})(\phi_{1t} + \phi_1\phi_{0u} - \xi_0\phi_{1x} + \phi_0\phi_{1u} + \phi_0)\big]\\
&\times\big(u-\xi_0^2\big)^2\\
&+\big(u-\xi_0^2\big)(2\xi_0\xi_{1t} + 2\xi_0\phi_0\xi_{1u} + 2u\xi_{1x} - \phi_1)(\phi_{0t} + \phi_0\phi_{0u} - \xi_0\phi_{0x})\\
&+(\phi_{0t} + \phi_0\phi_{0u} - \xi_0\phi_{0x})\big(4\xi_0^2\xi_1\xi_{0t} + 2\xi_0\xi_1\phi_0 + 4u\xi_0\xi_1\xi_{0x}\big) = 0
\end{aligned} \tag{14a}$$

$$\begin{aligned}
\big[&2\xi_{0t}\xi_{1x} - 2\xi_0\phi_{1tu} - 2\xi_0\phi_0\phi_{1uu} - \xi_{1tt} - 2\phi_0\xi_{1tu} - 2\xi_{1t}\phi_{0u} + 2\xi_{0x}\xi_{1t}\\
&-2\xi_{1u}\phi_0\phi_{0u} + 2\phi_0\xi_{1u}\xi_{0x} + 2\xi_0\xi_{1u}\phi_{0x} - \phi_0^2\xi_{1uu} - \xi_0\phi_{0u} - \xi_{0t} - \xi_{0u}\phi_0\\
&-2u\phi_{1xu} + u\xi_{1xx} - 2\phi_{1x} - 2\xi_1\phi_{0tu} + \xi_0\phi_{0u} - 2\xi_{0t}\phi_{1u}\big]\big(u-\xi_0^2\big)^2\\
&+\big[(2\xi_0\xi_{0t} - \phi_0 + 2u\xi_{0x})(\xi_0\xi_{1x} + \xi_1\xi_{0x} - 2\xi_1\phi_{0u} - 2\xi_0\phi_{1u} - \xi_{1t} - \phi_0\xi_{1u} - \xi_0)\\
&-(2\xi_0\xi_{1t} + 2\xi_0\phi_0\xi_{1u} - \phi_1 + 2u\xi_{1x})(2\phi_{0u}\xi_0 + \xi_{0t} + \xi_0\xi_{0x})\\
&+\big(2u\xi_{1u} - 2\xi_0^2\xi_{1u}\big)(\phi_{0t} + \phi_0\phi_{0u} - \xi_0\phi_{0x})\big]\big(u-\xi_0^2\big)\\
&-(2\xi_0\phi_{0u} + \xi_{0t} + \xi_0\xi_{0x})\big(4\xi_0^2\xi_{0t}\xi_1 + 2\xi_0\xi_1\phi_0 + 4u\xi_0\xi_{0x}\xi_1\big) = 0
\end{aligned} \tag{14b}$$

$$\begin{aligned}
\big[&\xi_0^2\phi_{1uu} + \phi_{1u} + 2\xi_0\xi_{1tu} + 2\xi_{0u}\xi_{1t} + 2\xi_{0u}\phi_0\xi_{1u} + 2\xi_0\xi_{1u}\phi_{0u} - 2\xi_0\xi_{1u}\xi_{0x}\\
&+2\xi_0\phi_0\xi_{1uu} + \xi_0\xi_{0u} - u\phi_{1uu} + 2u\xi_{1xu} - 2\phi_{1u} + 2\xi_{1x} + 2\xi_{0t}\xi_{1u}\big]\big(u-\xi_0^2\big)^2\\
&+\big[2\xi_0\xi_{1u}(2\xi_0\xi_{0t} - \phi_0 + 2u\xi_{0x}) - (2\xi_0\phi_{0u} + \xi_{0t} + \xi_0\xi_{0x})(2u\xi_{1u} - 2\xi_0^2\xi_{1u})\\
&-2\xi_0\xi_{1t} - 2\xi_0\phi_0\xi_{1u} + \phi_1 - 2u\xi_{1x}\big]\big(u-\xi_0^2\big)\\
&-4\xi_0^2\xi_1\xi_{0t} - 2\xi_0\xi_1\phi_0 - 4u\xi_0\xi_{0x}\xi_1 = 0
\end{aligned} \tag{14c}$$

$$\big(u-\xi_0^2\big)\xi_{1uu} - (2\xi_0\xi_{0u} + 1)\xi_{1u} = 0 \tag{14d}$$

where ξ_0 and ϕ_0 satisfy the overdetermined system (13) given in [20]. From system (14) and [20], we notice that the approximate conditional symmetries are inherited from the conditional symmetries of the unperturbed equation

$$u_{tt} = (uu_x)_x. \tag{15}$$

It is in general, not possible to obtain all solutions of the overdetermined system for the conditional symmetries of equation (15). Some special solutions are obtained in [20]. We list

Table 1. Conditional symmetries of equation (15) for $\tau_0 = 1$ (see [20]).

$V_{1,1} = \frac{\partial}{\partial t} + \sqrt{u}\frac{\partial}{\partial x} + c_1\frac{\partial}{\partial u}$
$V_{1,2} = \frac{\partial}{\partial t} - \sqrt{u}\frac{\partial}{\partial x} + c_1\frac{\partial}{\partial u}$
$V_{1,3} = \frac{\partial}{\partial t} + c_1 t\frac{\partial}{\partial x} + 2c_1^2 t\frac{\partial}{\partial u}$
$V_{1,4} = \frac{\partial}{\partial t} + a(t)u\frac{\partial}{\partial u} \qquad a'' + aa' - a^3 = 0$
$V_{1,5} = \frac{\partial}{\partial t} - \frac{x}{t^2}\frac{\partial}{\partial x} + \frac{2t^2u - 6x^2}{t^3}\frac{\partial}{\partial u}$
$V_{1,6} = \frac{\partial}{\partial t} + xt^{-1}\frac{\partial}{\partial x} + t^{-3}(x^2 - t^2u)\frac{\partial}{\partial u}$
$V_{1,7} = \frac{\partial}{\partial t} + t^{-1}u\frac{\partial}{\partial u}$
$V_{1,8} = t\frac{\partial}{\partial t} + \frac{1}{2}(5 - \sqrt{13})x\frac{\partial}{\partial x} + (3 - \sqrt{13})u\frac{\partial}{\partial u}$
$V_{1,9} = t\frac{\partial}{\partial t} + \frac{1}{2}(5 + \sqrt{13})x\frac{\partial}{\partial x} + (3 + \sqrt{13})u\frac{\partial}{\partial u}$

the corresponding results in table 1, where only the conditional symmetries $V_{1,3}$ and $V_{1,4}$ lead to nontrivial invariant solutions. The first two $V_{1,1}$ and $V_{1,2}$ give the simple solution $u = c_{11}t + c_{12}$, where c_{11}, c_{12} are constants. The other conditional symmetries lead to conditional invariant solutions which coincide with the second case.

We show by means of one example $V_{1,3}$ how one can obtain the approximate conditional symmetries and the associated approximate conditional invariant solutions.

For the conditional symmetry $V_{1,3}$,

$$\xi_0 = c_1 t \qquad \phi_0 = 2c_1^2 t \qquad c_1 = \text{constant}. \tag{16}$$

Substituting (16) into the system (14) and solving them for ξ_1 and ϕ_1, we have

$$\begin{aligned} \xi_1 &= -\tfrac{1}{5}x + b_{11}t + b_{12} \\ \phi_1 &= -\tfrac{2}{5}c_1x^2 - c_1^2t^2 + 4c_1b_{11}t + 2c_1b_{12} \end{aligned} \tag{17}$$

where $b_{ij}, i, j = 1, 2$ are constants.

By solving the characteristic equations

$$\frac{\mathrm{d}t}{1} = \frac{\mathrm{d}x}{c_1t + \epsilon\left(-\frac{1}{5}x + b_{11}t + b_{12}\right)} = \frac{\mathrm{d}u}{2c_1^2t + \epsilon\left(-\frac{2}{5}c_1x - c_1^2t^2 + 4c_1b_{11}t + 2c_1b_{12}\right)} \tag{18}$$

we obtain an approximate conditional invariant solution in the first order of precision, namely

$$u = c_1t^2 + f(\lambda) + \epsilon\left[-\tfrac{2}{5}c_1tx - \tfrac{1}{5}c_1^2t^3 + 2c_1b_{11}t^2 + 2c_1b_{12}t + g(\lambda)\right] \tag{19}$$

where λ is

$$\lambda = \mathrm{e}^{\frac{1}{5}\epsilon t}x - \tfrac{1}{2}c_1t^2 - \epsilon\left(\tfrac{1}{2}b_{11}t^2 + b_{12}t + \tfrac{1}{15}c_1t^3\right)$$

and $f(\lambda)$ and $g(\lambda)$ satisfy the system of ODEs

$$\begin{aligned} (ff_\lambda)_\lambda &= 2c_1^2 - c_1f_\lambda \\ fg_\lambda + f_\lambda g + c_1g &= 4c_1b_{11}\lambda - b_{11}f + b_{21}. \end{aligned} \tag{20}$$

For $b_{11} = 0$ and $b_{21} = 2c_1^2$, system (20) admits the explicit exact solution

$$
\begin{aligned}
f(\lambda) &= -\lambda + \frac{2^{2/3}\lambda^2}{(\lambda^3 + a_1^3 + a_1^{3/2}\sqrt{8\lambda^3 + a^3})^{1/3}} + \frac{\big(4\lambda^3 + a_1^3 + a_1^{3/2}\sqrt{8\lambda^3 + a_1^3}\big)^{1/3}}{2^{2/3}} \\
g(\lambda) &= -1 + \lambda(\lambda^3 + a_1^3 + a_1^{3/2}(8\lambda^3 + a_1^3)^{1/2})^{-4/3} \\
&\qquad \times\big[2^{5/2}a_1^3 + 2^{2/3}\lambda^3 + 2^{2/3}a_1^{3/2}(8\lambda^3 + a_1^3)^{-1/2}(4\lambda^3 - \tfrac{2}{3}a_1\lambda^2 - \tfrac{1}{3}a_1^2\lambda)\big] \\
&\qquad +\tfrac{1}{3}2^{-2/3}\big[4\lambda^3 + a_1^3 + a_1^{3/2}(8\lambda^3 + a_1^3)^{1/2}\big]^{-2/3} \\
&\qquad \times\big[12\lambda^2 + a_1^{3/2}(8\lambda^3 + a_1^3)^{-1/2}(12\lambda^2 + 2a_1\lambda + a_1^2)\big]
\end{aligned}
$$

where a_1 is a constant. This solution cannot be obtained by the approximate classical method.

For the conditional symmetry $V_{1,4}$, we find two types of solutions.

(1)

$$
\begin{aligned}
&\xi_0 = 0 \qquad \phi_0 = u/t \qquad \xi_1 = d_{11}t^2 + d_{12}t^{-2} \qquad \phi_1 = \big(d_{21}t^2 + d_{22}t^{-2} - \tfrac{1}{2}\big)u \\
&u = t(f_0\lambda + f_1)^{1/2} + \epsilon\big[tg(\lambda) + \big(\tfrac{1}{3}d_{21}t^4 - \tfrac{1}{2}t^2 - d_{22}\big)(f_0\lambda + f_1)^{1/2}\big]
\end{aligned}
\tag{21}
$$

where

$$
\begin{aligned}
\lambda &= x - \big(\tfrac{1}{3}d_{11}t^3 - d_{12}t^{-1}\big)\epsilon \\
g &= \frac{16d_{21}}{15f_0^2}(f_0\lambda + f_1)^2 - \frac{8d_{11}}{3f_0}(f_0\lambda + f_1) + (g_0\lambda + g_1)(f_0\lambda + f_1)^{-1/2}.
\end{aligned}
$$

In the above and hereafter f_i and g_i, b_{ij}, c_{ij} and d_{ij}, $i, j = 0, 1, 2$ are constants.

(2)

$$
\begin{aligned}
&\xi_0 = 0 \qquad \phi_0 = -2u/t \qquad \xi_1 = (d_{11}t^4 + d_{12}t^{-1})x + d_{41}t^4 + d_{42}t^{-1} \\
&\phi_1 = \big(d_{31}t^5 + 2d_{11}t^4 + 2d_{12}t^{-1} + d_{32}t^{-2} - \tfrac{1}{5}\big)u \\
&u = t^{-2}f(\lambda) + \epsilon\big[t^{-2}g(\lambda) + \big(\tfrac{1}{6}d_{31}t^4 + \tfrac{2}{5}d_{11}t^3 - d_{32}t^{-3} + 2d_{12}t^{-2}\ln t - \tfrac{1}{5}t^{-1}\big)f(\lambda)\big]
\end{aligned}
\tag{22}
$$

where

$$
\lambda = x - \epsilon\big[\big(\tfrac{1}{5}d_{11}t^5 + d_{12}\ln t\big)x + \tfrac{1}{5}d_{41}t^5 + d_{42}\ln t\big]
$$

and $f(\lambda)$ and $g(\lambda)$ satisfy

$$
\begin{aligned}
&ff_{\lambda\lambda} + f_\lambda^2 = 6f \\
&gf_{\lambda\lambda} + 2f_\lambda g_\lambda + fg_{\lambda\lambda} = (5d_{12}\lambda + d_{42})f_\lambda + 6g - 10d_{12}f.
\end{aligned}
\tag{23}
$$

It is difficult to determine the general solution of system (23). However, for $d_{12} = d_{42} = 0$, we can obtain the general solution of system (23) given by

$$
\begin{aligned}
&\int^{f(\lambda)} \frac{\mathrm{d}s}{\sqrt{4s^2 + f_0s^{-2}}} = \lambda + f_1 \\
&g(\lambda) = g_0\int^{\lambda} \frac{f^4(s)}{4f^3(s) + f_0}\,\mathrm{d}s + g_1f_\lambda.
\end{aligned}
$$

Case 2. $\tau_0 = 0, \xi_0 = 1, \xi_1 = 0.$

In this case, the determining equations for τ_1 and ϕ_1 are

$$\begin{aligned}&\phi_{1tt} - 4\phi_0\phi_1\phi_{0u} - 3\phi_0\phi_{1x} + \tau_1\phi_0\phi_{0t} + 2u\phi_0\tau_{1u}\phi_{0t} - 3\phi_{0x}\phi_1 - 2\phi_0^2\phi_{1u} + \phi_{0t}\\ &\quad -2\tau_{1t}\left(u\phi_{0x} + u\phi_0\phi_{0u} + \phi_0^2\right) - 2u\phi_1\phi_{0xu} - u\phi_{1xx} - 2u\phi_0\phi_{1xu}\\ &\quad +2u\tau_{1x}\phi_{0t} = 0 \qquad (24a)\end{aligned}$$

$$\begin{aligned}&4\tau_1\phi_0\phi_{0u} - 2\phi_0^2\tau_{1u} + \phi_0\tau_{1x} + 2\tau_1\phi_{0x} + 2\phi_{1tu} - \tau_{1tt} - 2u\tau_{1u}\phi_{0x} + 2u\tau_1\phi_{0xu}\\ &\quad +2u\phi_0\phi_{0uu}\tau_1 + 2u\phi_{0u}\tau_{1x} + u\tau_{1xx} + 2u\phi_0\tau_{1xu} + u\phi_0^2\tau_{1uu} = 0 \qquad (24b)\end{aligned}$$

$$\phi_{1uu} - 2\tau_{1tu} = 0 \qquad (24c)$$

$$\tau_{1uu} = 0 \qquad (24d)$$

where ϕ_0 is determined by equations (15) and (16) in [20]. For this case, we obtain some approximate conditional symmetries and approximate conditional invariant solutions as listed below.

(3)

$$\begin{aligned}&\tau_1 = 0 \qquad \phi_0 = b_{21}t + b_{22}\\ &\phi_1 = (c_{11}x + c_{12})u - \tfrac{5}{2}c_{11}(b_{21}t + b_{22})x^2 + d_1(t)x + d_2(t)\\ &u = (b_{21}t + b_{22})x + h(t) + \epsilon[((b_{21}t + b_{22}x) + h(t))(\tfrac{1}{2}c_{11}x^2 + c_{12}x)\\ &\qquad -c_{11}(b_{21}t + b_{22})x^3 + \tfrac{1}{2}(d_1(t) - c_{12}b_{22} - c_{12}b_{21}t)x^2 + d_2(t)x + m(t)]\end{aligned} \qquad (25)$$

where

$$\begin{aligned}d_1(t) &= -\tfrac{13}{12}c_{11}b_{21}^2t^4 - \tfrac{13}{3}c_{11}b_{21}b_{22}t^3 - \tfrac{13}{2}c_{11}b_{22}^2t^2 + d_{11}t + d_{12}\\ d_2(t) &= -\tfrac{13}{168}c_{11}b_{21}^3t^7 - \tfrac{13}{24}c_{11}b_{21}^2b_{22}t^6 - \tfrac{13}{8}c_{11}b_{21}b_{22}^2t^5\\ &\quad +\tfrac{1}{12}\left(3d_{11}b_{21} - \tfrac{39}{2}c_{11}b_{22}^3 + 2c_{12}b_{21}^2\right)t^4\\ &\quad +\tfrac{1}{6}(3b_{21}d_{12} + 3d_{11}b_{22} + 4c_{12}b_{21}b_{22})t^3\\ &\quad +\tfrac{1}{2}\left(3d_{12}b_{22} + 2c_{12}b_{22}^2 - b_{21}\right)t^2 + d_{21}t + d_{22}\\ h(t) &= \tfrac{1}{12}b_{21}^2(t + t_0)^4 + b_{31}t + b_{32}\\ m(t) &= -\tfrac{1}{378}c_{11}(t + t_0)^{10} + \tfrac{1}{504}(7d_{11} + 7c_{12} - 11c_{11}b_{31})(t + t_0)^7\\ &\quad +\tfrac{1}{360}(13d_{12} - 11c_{11}b_{32})(t + t_0)^6 - \tfrac{1}{15}(t + t_0)^5\\ &\quad +\tfrac{1}{12}(c_{11}b_{31}^2 + 3c_{12}b_{31} + d_{11}b_{31} + 2d_{21})(t + t_0)^4\\ &\quad +\tfrac{1}{6}(2c_{11}b_{31}b_{32} + 3c_{12}b_{32} + b_{31}d_{12} + d_{11}b_{32} + 2d_{22})(t + t_0)^3\\ &\quad +\tfrac{1}{2}(c_{11}b_{32}^2 + d_{12}b_{32} - b_{31})t^2 + d_{31}t + d_{32},\ b_{22} = b_{21}t_0.\end{aligned}$$

(4)

$$\begin{aligned}&\tau_1 = 0 \qquad \phi_0 = 2t^{-2}x + b_{21}t^3 + b_{22}t^{-2}\\ &\phi_1 = (d_{11}t^4 + d_{12}t^{-3} - \tfrac{2}{5}t^{-1})x + \tfrac{1}{22}b_{21}d_{11}t^9 + (\tfrac{1}{2}b_{22}d_{11} - \tfrac{7}{10}b_{21})t^4\\ &\qquad +d_{21}t^3 - \tfrac{3}{4}b_{21}d_{12}t^2 - \tfrac{1}{5}b_{22}t^{-1} + d_{22}t^{-2} + \tfrac{1}{2}b_{22}d_{12}t^{-3},\\ &u = t^{-2}x^2 + (b_{21}t^3 + b_{22}t^{-2})x + h(t) + \epsilon\left\{\tfrac{1}{2}(d_{11}t^4 - \tfrac{2}{5}t^{-1} + d_{12}t^{-3})x^2\right.\\ &\qquad +[\tfrac{1}{22}d_{11}b_{21}t^9 + (\tfrac{1}{2}d_{11}b_{22} - \tfrac{7}{10}b_{21})t^4 + d_{21}t^3 - \tfrac{3}{4}b_{21}d_{12}t^2 - \tfrac{1}{5}b_{22}t^{-1}\\ &\qquad \left.+d_{22}t^{-2} + \tfrac{1}{2}b_{22}d_{12}t^{-3}]x + m(t)\right\}\end{aligned} \qquad (26)$$

where

$$h(t) = \tfrac{1}{54}b_{21}^2 t^8 + \tfrac{1}{2}b_{21}b_{22}t^3 + b_{24}t^2 + b_{23}t^{-1} + \tfrac{1}{4}b_{22}^2 t^{-2}$$

$$\begin{aligned} m(t) = {} & \tfrac{13}{21\,384}d_{11}b_{21}^2 t^{14} + \left(\tfrac{1}{44}b_{21}d_{22}d_{11} - \tfrac{1}{45}b_{21}^2\right)t^9 \\ & + \tfrac{1}{54}(d_{11}b_{24} + 2b_{21}d_{21})t^8 - \tfrac{1}{27}t^7 + \tfrac{1}{8}d_{11}b_{23}t^5 + \left(\tfrac{1}{8}d_{11}b_{22}^2 - \tfrac{7}{20}b_{21}b_{22}\right)t^4 \\ & + \left(\tfrac{1}{2}b_{21}d_{22} + \tfrac{1}{2}b_{22}d_{21} - \tfrac{3}{5}b_{24}\right)t^3 + b_{25}t^2 - \tfrac{1}{2}d_{12}b_{24}t \\ & + b_{26}t^{-1} + \tfrac{1}{4}(d_{12}b_{23} + 2b_{22}d_{22})t^{-2} + \tfrac{1}{8}d_{12}b_{22}^2 t^{-3} - \tfrac{3}{10}b_{23}. \end{aligned}$$

(5)

$$\begin{aligned} & \tau_1 = 0 \qquad \phi_0 = 2u/x \\ & \phi_1 = (c_{11}x^{-2} + c_{12}x^{-7})u + (d_{11}x^{-2} + d_{12}x^{-3})t + d_{21}x^{-2} + d_{22}x^{-3} \\ & u = x^2 h(t) + \epsilon\big[(-c_{11}x - \tfrac{1}{6}c_{12}x^{-4})h(t) \\ & \qquad -(\tfrac{1}{3}d_{11}x^{-1} + \tfrac{1}{4}d_{12}x^{-2})t - \tfrac{1}{3}d_{21}x^{-1} - \tfrac{1}{4}d_{22}x^{-2} + m(t)x^2\big] \end{aligned} \tag{27}$$

where $h(t)$ and $m(t)$ are given in terms of Weierstrauss elliptic functions

$$h(t) = \wp(t + t_0, 0, t_1)$$

$$m(t) = d_{31}h' + d_{32}h' \int^t (h')^{-2}\,\mathrm{d}t.$$

(6)

$$\begin{aligned} & \tau_1 = 0 \qquad \phi_0 = u/(2x) \\ & \phi_1 = (c_{11}x^{1/2} + c_{12}x^{-2})u + (d_{11}x^{-3/2} + d_{12}x)t + d_{21}x^{-3/2} + d_{22}x \\ & u = x^{1/2}(c_{31}t + c_{32}) + \epsilon\big[(\tfrac{2}{3}c_{11}x^2 - c_{12}x^{-1/2})(c_{31}t + c_{32}) \\ & \qquad +(\tfrac{2}{3}d_{12}x^2 - d_{11}x^{-1/2})t - d_{21}x^{-1/2} + \tfrac{2}{3}d_{22}x^2 \\ & \qquad +(\tfrac{5}{16}(c_{11} + d_{12})t^4 + \tfrac{5}{8}d_{22}t^3 + d_{31}t + d_{32})x^{1/2}\big]. \end{aligned} \tag{28}$$

(7)

$$\begin{aligned} & \tau_1 = 0 \qquad \phi_0 = u/(2x) + \tfrac{3}{2}t^{-2}x \\ & \phi_1 = c_{12}x^{-2}u + (d_{11}t^{3/2} + d_{12}t^{-1/2})x^{-3/2} + (d_{21}t^4 + d_{22}t^{-3} - \tfrac{3}{10}t^{-1})x - 3c_{12}t^{-2} \\ & u = h(t)x^{1/2} + t^{-2}x^2 + \epsilon\big[\tfrac{2}{3}\big(d_{21}t^4 + d_{22}t^{-3} - \tfrac{3}{10}t^{-1}\big)x^2 \\ & \qquad -4c_{12}t^{-2}x + m(t)x^{1/2} - \big(d_{11}t^{3/2} + d_{12}t^{-1/2} + c_{12}h(t)\big)x^{-1/2}\big] \end{aligned} \tag{29}$$

where

$$h = c_{31}t^{-3/2} + c_{32}t^{5/2}$$

$$\begin{aligned} m = {} & \tfrac{1}{24}c_{32}d_{21}t^{17/2} + \tfrac{5}{24}c_{31}d_{21}t^{9/2} - \tfrac{13}{20}c_{32}t^{7/2} + d_{32}t^{5/2} \\ & -\tfrac{5}{6}c_{32}d_{22}t^{3/2} - \tfrac{1}{4}c_{31}t^{-1/2} + d_{31}t^{-3/2} + \tfrac{1}{2}c_{31}d_{22}t^{-5/2}. \end{aligned}$$

4. Nonlinear heat equation

In this section, we turn to the approximate conditional symmetries of the nonlinear heat equation

$$u_t = u_{xx} - u^3 + \epsilon u. \tag{30}$$

Equation (30) arises in several important physical applications including microwave heating and chemical reactions [21, 22]. The conditional symmetries of the unperturbed equation of equation (30), namely

$$u_t = u_{xx} - u^3 \tag{31}$$

have been discussed in [6, 23] and are listed in table 2. As for the nonlinear wave equation (13), we consider two cases.

Table 2. Conditional symmetries of equation (31) (see [6, 23]).

Number	ξ_0	τ_0	ϕ_0
$V_{4,1}$	$\frac{3}{2}\sqrt{2}u$	1	$-\frac{3}{2}u^3$
$V_{4,2}$	$-\dfrac{3}{x+k_1}$	1	$-\dfrac{3}{(x+k_1)^2}$
$V_{4,3}$	1	0	$\frac{1}{2}\sqrt{2}u^2$
$V_{4,4}$	1	0	$\dfrac{\sqrt{2}}{2}u^2 + \dfrac{u}{x+k_1}$
$V_{4,5}$	1	0	$\dfrac{\sqrt{2}}{2}u^2 + \dfrac{2x+k_1}{x^2+k_1x-6t}u + \dfrac{2\sqrt{2}}{x^2+k_1x-6t}$

Case 1. $\tau \neq 0$, $\tau_0 = 1$, $\tau_1 = 0$.

The determining equations for ξ_1 and ϕ_1 are

$$\phi_{1t} - \phi_{1xx} - u^3\phi_{1u} + 2u^3\xi_{1x} + 3u^2\phi_1 - \phi_0 + 2\xi_{1x}\phi_0 + 2\xi_{0x}\phi_1 - 2u\xi_{0x} + u\phi_{0u} = 0 \tag{32a}$$

$$\xi_{1xx} - 2\phi_{1xu} - 2\xi_0\xi_{1x} + 2\phi_0\xi_{1u} + 3u^3\xi_{1u} - \xi_{1t} - 2\xi_1\xi_{0x} + 2\phi_1\xi_{0u} - 3u\xi_{0u} = 0 \tag{32b}$$

$$2\xi_{1xu} - \phi_{1uu} - 2\xi_0\xi_{1u} - 2\xi_1\xi_{0u} = 0 \tag{32c}$$

$$\xi_{1uu} = 0. \tag{32d}$$

The zeroth-order determining equations for ξ_0 and ϕ_0 are given in [6] and the solutions are presented in the first two entries of table 2, so we consider two cases.

(1)

$$\xi_0 = \tfrac{3}{2}\sqrt{2}u \qquad \phi_0 = -\tfrac{3}{2}u^3.$$

Solving system (32), we find

$$\xi_1 = 0 \qquad \phi_1 = \tfrac{3}{2}u$$

and an approximate conditional symmetry of equation (30)

$$V = V_0 + \epsilon V_1 = \frac{\partial}{\partial t} + \frac{3\sqrt{2}}{2}u\frac{\partial}{\partial x} - \frac{3}{2}(u^2 - \epsilon u)\frac{\partial}{\partial u}. \tag{33}$$

Actually, the approximate conditional symmetry (33) is a first-order truncated symmetry of equation (30), namely it is an exact conditional symmetry of equation (30).
An exact solution of the unperturbed equation (31) corresponding to V_0 is [6, 23]

$$u = \frac{x + k_2}{\frac{3}{2}\sqrt{2}(t + k_1) + \frac{1}{4}\sqrt{2}(x + k_2)^2} \tag{34}$$

where k_i, $i = 1, 2$, are constants. Since the approximate symmetry is a truncated symmetry, the corresponding approximate conditional invariant solution is an exact solution of equation (30). In terms of the sign of ϵ, we obtain solutions of equation (30) given by

(A) $\epsilon > 0$:

$$u_1 = \frac{\sqrt{\epsilon}\left[\mathrm{e}^{\sqrt{\frac{1}{2}\epsilon}(x+\frac{3}{2}\sqrt{2\epsilon}\,t)} - k_1\mathrm{e}^{-\sqrt{\frac{1}{2}\epsilon}\,(x-\frac{3}{2}\sqrt{2\epsilon}\,t)}\right]}{\mathrm{e}^{\sqrt{\frac{1}{2}\epsilon}\left(x+\frac{3}{2}\sqrt{2\epsilon}\,t\right)} + k_1\mathrm{e}^{-\sqrt{\frac{1}{2}\epsilon}\left(x-\frac{3}{2}\sqrt{2\epsilon}\,t\right)} + k_2}. \tag{35}$$

(B) $\epsilon < 0$:

$$u_2 = \frac{\sqrt{-\epsilon}\left(k_1\cos\sqrt{-\frac{1}{2}\epsilon}\,x - \sin\sqrt{-\frac{1}{2}\epsilon x}\right)}{\cos\sqrt{-\frac{1}{2}\epsilon}\,x + k_1\sin\sqrt{-\frac{1}{2}\epsilon}\,x + k_2\mathrm{e}^{-\frac{3}{2}t}}. \tag{36}$$

It is easy to see that solutions u_1 and u_2 converge to zero uniformly as ϵ approaches zero—not to the solution (34) of the unperturbed equation (31), which shows that the solution (34) is unstable.

(2)

$$\xi_0 = -\frac{3}{(x + k_1)} \qquad \phi_0 = -\frac{3u}{(x + k_1)^2}.$$

In this case, we find an approximate conditional symmetry inherited from the symmetry $V_{4,2}$,

$$V_2 = \frac{\partial}{\partial t} + \left[-\frac{3}{x + k_1} + \epsilon\left(\frac{b_{11}}{(x + k_1)^2} - \frac{1}{2}(x + k_1)\right)\right]\frac{\partial}{\partial x} + \left[-\frac{3}{(x + k_1)^2} + \epsilon\left(\frac{2b_{11}}{(x + k_1)^3} + \frac{1}{2}\right)\right]u\frac{\partial}{\partial u} \tag{37}$$

and the associated approximate conditional invariant solution of equation (30)

$$u = \tfrac{1}{3}(x + k_1)f(\lambda) + \epsilon\left[\left(\tfrac{1}{9}b_{11} - \tfrac{1}{18}(x + k_1)^3\right)f(\lambda) + \tfrac{1}{3}(x + k_1)g(\lambda)\right] \tag{38}$$

where

$$\lambda = t + \tfrac{1}{6}(x + k_1)^2 + \epsilon\left[\tfrac{1}{9}b_{11}(x + k_1) - \tfrac{1}{72}(x + k_1)^4\right]$$

$$f = \sqrt{2}\,\mathrm{d}s(\lambda, \tfrac{1}{2}\sqrt{2})$$

$$g = c_{11}f_\lambda\int f_\lambda^{-2}\,\mathrm{d}\lambda + c_{21}f_\lambda + \tfrac{9}{2}f_\lambda\int^\lambda f(\lambda)f_\lambda^{-2}\,\mathrm{d}\lambda$$

and $\mathrm{d}s(\lambda, k)$ is the Jacobi elliptic function which satisfies

$$\left(\frac{\mathrm{d}\eta}{\mathrm{d}\lambda}\right)^2 = k^2(k^2 - 1) + (2k^2 - 1)\lambda^2 + \lambda^4.$$

Case 2. $\tau_0 = 0, \xi_0 = 1, \xi_1 = 0.$

The system of determining equations for τ_1, ϕ_0 and ϕ_1 is

$$\phi_{0t} - u^3\phi_{0u} - \phi_{0xx} - 2\phi_0\phi_{0xu} + 3u^2\phi_0 - \phi_0^2\phi_{0uu} = 0 \tag{39a}$$

$$\begin{aligned}
&\phi_{1t} - \phi_{1xx} - 2\phi_0\phi_{1xu} - \phi_0^2\phi_{1uu} - 2\phi_1\phi_{0xu} - 2\phi_0\phi_1\phi_{0uu} - u^3\phi_{1u} \\
&\quad + u\phi_{0u} + 3u^2\phi_1 - \phi_0 + \tau_{1xx}\left(\phi_{0x} + \phi_0\phi_{0u} - u^3\right) \\
&\quad + \tau_{1u}\left(2\phi_0\phi_{0t} + 2\phi_0\phi_{0x}\phi_{0u} + 2\phi_0\phi_{0u}^2 + u^3\phi_{0x} - u^3\phi_0\phi_{0u} - u^3\right) \\
&\quad + 2\tau_{1xu}\left(\phi_0\phi_{0x} + \phi_0^2\phi_{0u} - u^3\phi_0\right) + 2\tau_{1x}\left(\phi_{0t} + \phi_0\phi_{0u}^2 + \phi_{0x}\phi_{0u} - u^3\phi_{0u}\right) \\
&\quad - \tau_{1t}\left(\phi_{0x} + \phi_0\phi_{0u} - u^3\right) + \tau_{1uu}\left(\phi_0^2\phi_{0x} + \phi_0^3\phi_{0u} - u^3\phi_0^2\right) \\
&\quad + 2\tau_1\left(\phi_{0x}\phi_{0xu} + \phi_0\phi_{0u}\phi_{0xu} - 2u^3\phi_{0xu} + \phi_0\phi_{0x}\phi_{0uu}\right. \\
&\quad \left. + 2\phi_0^2\phi_{0u}\phi_{0uu} - 2u^3\phi_0\phi_{0uu}\right) = 0.
\end{aligned} \tag{39b}$$

The system (39) with two equations involves three dependent variables, ϕ_0, ϕ_1 and τ_1, so it is not possible to find all solutions. Special solutions will be considered. Set

$$\phi_0 = A(x,t)u^2 + B(x,t)u + C(x,t). \tag{40}$$

The substitution of (40) into (39a) implies that A, B and C satisfy

$$\begin{aligned}
&2A^2 - 1 = 0 \\
&C - 2A(B_x + B^2) = 0 \\
&B_t - B_{xx} - 2BB_x - 4ABC = 0 \\
&C_t - C_{xx} - 2CB_x - 2AC^2 = 0.
\end{aligned} \tag{41}$$

Solutions of the system (41) are given by

(i) $\quad A = \dfrac{1}{2}\sqrt{2} \qquad B = C = 0$

(ii) $\quad A = \dfrac{1}{2}\sqrt{2} \qquad B = \dfrac{1}{x + b_2} \qquad C = 0$

(iii) $\quad A = \dfrac{1}{2}\sqrt{2} \qquad B = \dfrac{2x + b_1}{x^2 + b_1 x - 6t} \qquad C = \dfrac{2\sqrt{2}}{x^2 + b_1 x - 6t}.$

Solutions (ii) and (iii) were not given in [6]. Solution (i) yields the stationary solution of the unperturbed equation (31). Unfortunately, solutions (ii) and (iii) lead to the exact solution (34) of equation (31) again. To treat (39b), we restrict $\tau_1 = 0$ and $\phi_1 = a(x,t)u^2 + b(x,t)u + c(x,t)$. We consider two cases corresponding to $V_{4,3}$ and $V_{4,4}$.

(1)

$$\xi_0 = 1 \qquad \phi_0 = \tfrac{1}{2}\sqrt{2}u^2.$$

Substituting ϕ_0 and ϕ_1 into (39b), we find a, b, and c satisfying

$$a = 0 \qquad c = \sqrt{2}b_x - \tfrac{1}{2}\sqrt{2} \qquad b_t - b_{xx} = 0. \tag{42}$$

Two special solutions are considered.

(A) $b = 0$. We deduce an approximate conditional symmetry

$$V_3 = \frac{\partial}{\partial x} + \tfrac{1}{2}\sqrt{2}(u^2 - \epsilon)\frac{\partial}{\partial u} \tag{43}$$

which actually is a first-order truncated symmetry [6], so that V_3 gives exact solutions of equation (30).

If $\epsilon > 0$,

$$u = -\sqrt{\epsilon}\coth\tfrac{1}{2}\sqrt{2\epsilon}\,(x - 2\sqrt{2\epsilon}t). \tag{44}$$

If $\epsilon < 0$,

$$u = \sqrt{-\epsilon}\tan\tfrac{1}{2}\sqrt{-2\epsilon}\,(x - 2\sqrt{-2\epsilon}t). \tag{45}$$

Solutions (44) and (45) do not converge to the stationary solution of equation (31), so the stationary solution is also unstable.

(B) $b = \frac{1}{2}x$. We find an approximate stationary solution

$$u = \frac{1}{k_1 - \frac{1}{2}\sqrt{2}x} + (k_1 - \tfrac{1}{2}\sqrt{2}x)^{-2}(\tfrac{1}{4}k_1^2 x + \tfrac{3}{2}k_1 t - \tfrac{1}{12}\sqrt{2}x^3)\epsilon. \tag{46}$$

(2)

$$\xi_0 = 1 \qquad \phi_0 = \tfrac{1}{2}\sqrt{2}u^2 + \frac{u}{x + b_2}.$$

In this case, we obtain an approximate conditional symmetry

$$V_4 = \frac{\partial}{\partial x} + \left[\frac{\sqrt{2}}{2}u^2 + \frac{u}{x+b_2} + \epsilon\Big(\big(6b_{11}t(x+b_2)^{-2} + \tfrac{1}{6}(x+b_2) + b_{11}\big)u \right.$$
$$\left. + 2\sqrt{2}b_{11}(x+b_2)^{-1}\Big)\right]\frac{\partial}{\partial u}. \tag{47}$$

For $b_{11} = 0$, the corresponding approximate conditional invariant solution is

$$u = (x+b_2)\big(-\tfrac{3}{2}\sqrt{2}t + b_{21} - \tfrac{1}{4}\sqrt{2}(x+b_2)^2\big)^{-1}$$
$$+\epsilon\big[\tfrac{1}{12}(x+b_2)^3\big(-\tfrac{3}{2}\sqrt{2}t + b_{21} - \tfrac{1}{4}\sqrt{2}(x+b_2)^2\big)^{-1}$$
$$-(x+b_2)\big(\tfrac{9}{8}\sqrt{2}t^2 - \tfrac{3}{2}b_{21}t + b_{22} - \tfrac{1}{96}\sqrt{2}(x+b_2)^4\big)$$
$$\times\big(-\tfrac{3}{2}\sqrt{2}t + b_{21} - \tfrac{1}{4}\sqrt{2}(x+b_2)^2\big)^{-2}\big]. \tag{48}$$

5. Discussion

In this paper, we extended the theory of the approximate classical symmetry method of Baikov *et al* [14, 15] to include conditional symmetries and this enables us to construct approximate conditional invariant solutions for nonlinear PDEs depending on a small parameter. In particular, for the nonlinear wave equation and the nonlinear heat equation considered, we constructed some approximate conditional invariant solutions. Just as the conditional invariant solutions cannot in general be obtained by the classical method, so too those solutions except for certain cases cannot be obtained by the approximate classical symmetry method. For nonlinear partial differential equations with a small parameter, we introduced the notion of a truncated symmetry. For the nonlinear heat equation (30), when it admits the truncated symmetries of first order, we cannot obtain approximate exact solutions. The stability property of exact solutions for unperturbed equations was also considered. As we have seen before, the overdetermined system for approximate conditional symmetries are more complicated than that for the ordinary conditional symmetries, so it will be of interest to develop computer algebra programs to solve the overdetermined system for approximate conditional symmetries. In particular, the algorithm of [24] could be useful for this purpose.

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